Algorithms and Data Structures
Spring 2018
Exercises for Units 23 and 24

1. Suppose you are given a directed graph $G = (V, E)$, two vertices s and t, a capacity function $c : E \to \mathbb{R}^+$, and another function $f : E \to \mathbb{R}$. Describe an algorithm to determine whether f is a maximum (s, t)-flow in G.

2. Let (S, T) and (S', T') be minimum (s, t)-cuts in some flow network G. Prove that $(S \cap S', T \cup T')$ and $(S \cup S', T \cap T')$ are also minimum (s, t)-cuts in G.

3. Let $G = (V, E; s, t; c)$ be a capacitated network with source vertex s and sink vertex t. Assume that a maximum flow f has been computed.

   Now assume the capacity of an edge is changed.

   How difficult is it to compute from f a maximum flow of the changed network? (You may have to distinguish whether the capacity was increased or decreased.)

4. Prove the following two versions of Menger's theorem:

   (a) Let $G = (V, E)$ be an undirected graph, and let u and v be nonadjacent vertices in G
   The maximum number of pairwise edge disjoint (u, v)-paths in G equals the minimum number of edges from E whose deletion separates u and v.

   (b) Let $G = (V, E)$ be an undirected graph, and let u and v be nonadjacent vertices in G
   The maximum number of pairwise internal-vertex disjoint (u, v)-paths in G equals the minimum number of vertices from $V \setminus \{u, v\}$ whose deletion separates u and v.

5. Show that if all the capacities of a network are 1, then the maximum flow problem can be solved in time $O(n^{2/3}(n + m))$ or alternatively in time $O(m^{1/2}(n + m))$.

6. Consider the circulation problem on a network $G = (V, E)$ with edge capacities $c : E \to \mathbb{R}^+$ and node supplies $b : V \to \mathbb{R}$ with $b(V) = 0$.. Prove the following theorem:

   The circulation problem has a feasible solution iff for each $S \subset V$ we have $b(S) \leq c(S, V \setminus S)$.

7. We are given an $n \times m$ matrix M with nonnegative real entries. We want to round each entry M_{ij} to either $\lceil M_{ij} \rceil$ or $\lfloor M_{ij} \rfloor$ such that each row sum and each column sum is also rounded to a nearest integer. For example

   $\begin{pmatrix} 1.3 & 2.4 \\ 2.9 & 1.7 \end{pmatrix}$ could be rounded to $\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$ but not to $\begin{pmatrix} 2 & 3 \\ 3 & 1 \end{pmatrix}$

   since in the latter array the first row sums to 5, which is not a nearest integer of $3.7 = 1.3 + 2.4$.

   For a given matrix find such a rounding, if it exists. Does such a rounding always exist?

   Hint: Set up a flow problems with two special vertices s and t and one vertex for each row and one vertex for each column.