Problem 3

We need to prove that $f(x) \geq cx^p\big(1 + \int\limits_1^x \frac{g(u)}{u^{p+1}}du\big)$, for some constant $c \Rightarrow f(x) \in \Omega\big(x^p\big(1 + \int\limits_1^x \frac{g(u)}{u^{p+1}}du\big)\big)$.

We define ïndex of $x$äs number k, such that $\frac{x}{b^k} \leq x_0 < \frac{x}{b^{k+1}}$. Proof is by induction on k.

Base Case: k $= 0 \Rightarrow x \leq x_0$. We have to prove that $f(x) = h(x) \geq cx^p(\big(1+\int\limits_1^x \frac{g(u)}{u^{p+1}}du\big)$. Let $M = \max\{\frac{g(u)}{u^{p+1}} | 1 \leq u \leq x_0\} \Rightarrow \int\limits_1^x \frac{g(u)}{u^{p+1}}du \leq M(x-1) \leq M(x_0 - 1)$. Also, let $M' = \max(1, x_0{}^p)$.

Then, if we choose constant $c < \frac{d_1}{M'(1+M(x_0-1))}$, we will get that $h(x) \geq d_1 \geq cM'(1+M(x_0 - 1)) \geq cx^p\big(1 + \int\limits_1^x \frac{g(u)}{u^{p+1}}du\big)$.

Induction Step: $f(x) = af(\frac{x}{b}) + g(x) \geq ac(\frac{x}{b})^p\big(1+\int\limits_1^{\frac{x}{b}} \frac{g(u)}{u^{p+1}}du\big) + g(x) = cx^p\big(1+\int\limits_1^{\frac{x}{b}} \frac{g(u)}{u^{p+1}}du\big) + g(x)$ (1)

We know that $u^{p+1} \geq \min\big((\frac{x}{b})^{p+1}, x^{p+1}\big)$, for all $\frac{x}{b} \leq u \leq x$. Also, using the polynomial growth condition and by assuming that constant $c < \frac{\min\big((\frac{1}{b})^{p+1},1\big)}{c_2(1-\frac{1}{b})}$, we get that $cx^p\big(\int\limits_{\frac{x}{b}}^x \frac{g(u)}{u^{p+1}}du\big) \leq cx^p\big(\int\limits_{\frac{x}{b}}^x \frac{c_2 g(x)}{u^{p+1}}du\big) \leq cx^p(x - \frac{x}{b})\frac{c_2 g(x)}{\min\big((\frac{x}{b})^{p+1}, x^{p+1}\big)} = c\frac{c_2(1-\frac{1}{b})}{\min\big((\frac{1}{b})^{p+1},1\big)}g(x) \leq g(x)$. If we combine this inequality with the inequality (1) , we get that $f(x) \geq cx^p\big(1 + \int\limits_1^{\frac{x}{b}} \frac{g(u)}{u^{p+1}}du\big) + cx^p\big(\int\limits_{\frac{x}{b}}^x \frac{g(u)}{u^{p+1}}du\big) = cx^p\big(1 + \int\limits_1^x \frac{g(u)}{u^{p+1}}du\big)$.

Problem 1

Example Solutions for Assignment 2

Question 1

Akra-Bazzi Method

a) $g(n) = \sqrt{n}$
$a = 2, b = 4 \Rightarrow p = \frac{\log 2}{\log 4} = \frac{1}{2}$.

By the Akra-Bazzi theorem follows:

$$\begin{aligned}
T(n) &= \Theta(n^{\frac{1}{2}} \cdot (1 + \int_1^n \frac{\sqrt{u}}{u^{\frac{1}{2}+1}}\,\mathrm{d}u)) \\
&= \Theta(n^{\frac{1}{2}} \cdot (1 + \int_1^n \frac{u^{\frac{1}{2}}}{u^{\frac{3}{2}}}\,\mathrm{d}u)) \\
&= \Theta(n^{\frac{1}{2}} \cdot (1 + \int_1^n u^{-1}\,\mathrm{d}u)) \\
&= \Theta(n^{\frac{1}{2}} \cdot (1 + [\ln u]_1^n)) \\
&= \Theta(n^{\frac{1}{2}} \cdot (1 + \ln n - \ln 1)) \\
&= \Theta(n^{\frac{1}{2}} + n^{\frac{1}{2}} \cdot \ln n) \\
&= \Theta(\sqrt{n} \cdot \ln n)
\end{aligned}$$

b) $g(n) = n \cdot \log n$
$a = 3, b = 2 \Rightarrow p = \frac{\log 3}{\log 2} = \log 3$.

By the Akra-Bazzi theorem follows:

$$
\begin{aligned}
T(n) &= \Theta(n^{\log 3} \cdot (1 + \int_1^n \frac{u \cdot \log u}{u^{\log 3+1}} \,\mathrm{d}u)) \\
&= \Theta(n^{\log 3} \cdot (1 + \int_1^n \log u \cdot \frac{u^1}{u^{\log 3+1}} \,\mathrm{d}u)) \\
&= \Theta(n^{\log 3} \cdot (1 + \int_1^n \underbrace{\log u}_{v} \cdot \underbrace{u^{-\log 3}}_{u'} \mathrm{d}u)) \\
&= \Theta(n^{\log 3} \cdot (1 + \left[\underbrace{\frac{1}{-\log 3+1} u^{-\log 3+1}}_{u} \cdot \underbrace{\log u}_{v}\right]_1^n - \int_1^n \underbrace{\frac{1}{u \cdot \ln 2}}_{v'} \cdot \underbrace{\frac{1}{-\log 3+1} u^{-\log 3+1}}_{u} \mathrm{d}u)) \\
&= \Theta(n^{\log 3} \cdot (1 + \left[\frac{u^{1-\log 3}}{1-\log 3} \cdot \log u\right]_1^n - \frac{1}{\ln 2 \cdot (1-\log 3)} \cdot \int_1^n \frac{u^{1-\log 3}}{u} \,\mathrm{d}u)) \\
&= \Theta(n^{\log 3} \cdot (1 + \left[\frac{u^{1-\log 3}}{1-\log 3} \cdot \log u\right]_1^n - \frac{1}{\ln 2 \cdot (1-\log 3)} \cdot \int_1^n u^{-\log 3} \,\mathrm{d}u)) \\
&= \Theta(n^{\log 3} \cdot (1 + \left[\frac{u^{1-\log 3}}{1-\log 3} \cdot \log u\right]_1^n - \frac{1}{\ln 2 \cdot (1-\log 3)} \cdot \left[\frac{1}{1-\log 3} u^{1-\log 3}\right]_1^n)) \\
&= \Theta(n^{\log 3} \cdot (1 + \frac{n^{1-\log 3}}{1-\log 3} \cdot \log n - \frac{1}{\ln 2 \cdot (1-\log 3)} \cdot (\frac{n^{1-\log 3}}{1-\log 3} - \frac{1}{1-\log 3}))) \\
&= \Theta(n^{\log 3} \cdot (1 + \frac{n^{1-\log 3}}{1-\log 3} \cdot \log n - \frac{n^{1-\log 3}}{\ln 2 \cdot (1-\log 3)^2} + \frac{1}{\ln 2 \cdot (1-\log 3)^2})) \\
&= \Theta(n^{\log 3} + \frac{n \log n}{1-\log 3} - \frac{n}{\ln 2 \cdot (1-\log 3)^2} + \frac{n^{\log 3}}{\ln 2 \cdot (1-\log 3)^2}) \\
&= \Theta(n^{\log 3})
\end{aligned}
$$

c) $g(n) = \alpha$
$a = 1, b = 2 \Rightarrow p = \frac{\log 1}{\log 2} = 0.$
By the Akra-Bazzi theorem follows:

$$
\begin{aligned}
T(n) &= \Theta(n^0 \cdot (1 + \int_1^n \frac{\alpha}{u^{0+1}} \,\mathrm{d}u)) \\
&= \Theta(1 + \alpha \cdot \int_1^n \frac{1}{u} \,\mathrm{d}u) \\
&= \Theta(\alpha \cdot [\ln u]_1^n) \\
&= \Theta(\ln n)
\end{aligned}
$$

d) $g(n) = \frac{n^2}{\log n}$
$a = 4, b = 2 \Rightarrow p = \frac{\log 4}{\log 2} = 2.$

By the Akra-Bazzi theorem follows:

$$\begin{aligned} T(n) &= \Theta(n^2 \cdot (1 + \int_1^n \frac{\frac{u^2}{\log u}}{u^{2+1}} \, \mathrm{d}u)) \\ &= \Theta(n^2 \cdot (1 + \int_1^n \frac{u^2}{u^3 \cdot \log u} \, \mathrm{d}u)) \\ &= \Theta(n^2 \cdot (1 + \int_1^n \frac{1}{u \cdot \log u} \, \mathrm{d}u)) \end{aligned}$$

If we take a close look at the integral we got by applying the Akra-Bazzi method, we notice that the integrand goes to ∞ if u approaches the lower limit of 1. The integral doesn't exist! However, we notice that, since $T(n) = c \cdot n$ for $n \leq 4$, we don't alter $T(n)$ if we change the behaviour of $g(n)$ for $n \leq 4$. We choose

$$g(n) = \begin{cases} 8 & n \leq 4 \\ \frac{n^2}{\log n} & n > 4 \end{cases}$$

to get a function which is continous and defined for all real numbers. Now we can apply Akra-Bazzi successfully:

$$\begin{aligned} T(n) &= \Theta(n^2 \cdot (1 + \int_1^n \frac{g(u)}{u^{2+1}} \, \mathrm{d}u)) \\ &= \Theta(n^2 \cdot (1 + \int_1^4 \frac{g(u)}{u^3} \, \mathrm{d}u + \int_4^n \frac{g(u)}{u^3} \, \mathrm{d}u)) \\ &= \Theta(n^2 \cdot (1 + \underbrace{\int_1^4 \frac{8}{u^3} \, \mathrm{d}u}_{constant} + \int_4^n \frac{u^2}{u^3 \cdot \log u} \, \mathrm{d}u)) \\ &= \Theta(n^2 \cdot (1 + \int_4^n \frac{1}{\underbrace{u \cdot \log u}_{t := \log u, \frac{\mathrm{d}t}{\mathrm{d}u} = \frac{1}{\ln 2 \cdot u} \Rightarrow \mathrm{d}u = \ln 2 \cdot u \, \mathrm{d}t}} \, \mathrm{d}u)) \\ &= \Theta(n^2 \cdot (1 + \int_{\log 4}^{\log n} \frac{\ln 2 \cdot u}{u \cdot t} \, \mathrm{d}t)) \\ &= \Theta(n^2 \cdot (1 + \ln 2 \cdot \int_2^{\log n} \frac{1}{t} \, \mathrm{d}t)) \\ &= \Theta(n^2 \cdot (1 + \ln 2 \cdot [\ln t]_2^{\log n})) \\ &= \Theta(n^2 \cdot (1 + \ln 2 \cdot (\ln \log n - \ln 2))) \\ &= \Theta(n^2 + n^2 \cdot \ln 2 \cdot \ln \log n - n^2 \cdot \ln 2 \cdot \ln 2)) \\ &= \Theta(n^2 \cdot \ln \log n) \end{aligned}$$

If we think a bit more about it, we come to the conclusion that the lower limit of 1 for the integrals is somewhat arbitrary. The Akra-Bazzi theorem allows us to choose any natural number n_0 as separator between the two parts of the always partially defined function $T(n)$. We then can follow the same argument as shown in this example to change the lower limit of the integral to n_0.

Exercises for Unit 5

(2.) Notice that all conditions of Akkra-Bazi, other than the one containing an integral - this we will have to check, are satisfied by the assumptions of the Master Theorem.

$a < b^d$ $\Rightarrow p = \log_b a < d \cdot \log_b b = d$

$c := d - p$

$$\int_1^x \frac{f(u)}{u^{p+1}}\,du = \int_1^x u^{d-p-1}\,du = \int_1^x u^{c-1} = \frac{u^c}{c}\Big|_1^x = \frac{1}{c}x^c - \frac{1}{c}$$

$$= \Theta(x^c) = \Theta(x^{d-p}) \text{, so also finite}$$

$$\Rightarrow T(x) = \Theta(x^p(1 + x^{d-p})) = \Theta(x^d) = \Theta(f(x))$$

$$\Rightarrow T(n) = \Theta(n^d)$$

$a = b^d$ $\Rightarrow p = d$

$$\int_1^x \frac{f(u)}{u^{p+1}}\,du = \int_1^x \frac{1}{u}\,du = \log x = \Theta(\log x) \text{, so also finite}$$

$$\Rightarrow T(x) = \Theta(x^p(1 + \log x)) = \Theta(x^d \log x)$$

$$\Rightarrow T(n) = \Theta(n^d \log n)$$

$a > b^d$ $\Rightarrow p > d$, $c := d - p$

$$\int_1^x \frac{f(u)}{u^{p+1}}\,du = \int_1^x u^{+c-1}\,du = \frac{u^{+c}}{+c}\Big|_1^x = +\frac{1}{c}(x^{+c} - 1) = \Theta(x^{+c})$$

↳ so also finite

$$\Rightarrow T(x) = \Theta(x^p(1 + x^{+c})) = \Theta(x^p) = \Theta(x^{\log_b a})$$

$$\Rightarrow T(n) = \Theta(n^{\log_b a})$$