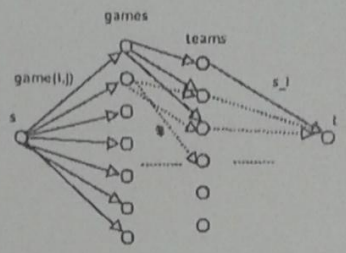
~~5 Exercise 5~~ = 2 Unit 27 Problem 1

Let's say we want to check if a team t_k can win. We can bound the maximal number of additional to $score(j)$ points any other team $t_j, j \neq k$ can get (still having t_k winning) as follows:

$$s_k = \sum_{j \neq k} games(k, j) \quad (1)$$

$$s_j = s_k + score(k) - score(j) - 1,\ j \neq k \quad (2)$$

Consider the following max flow network:



Connect s with $game(i, j)$ by an edge with capacity $games(i, j)$, $game(i, j)$ with teams i, j with capacity ∞ each team $t_i, i \neq k$ with t with capacity s_i.

We know that the team k can attain s_k, so we omit all the games with team k in the graph. By finding a maximal flow in the network, we can find out if the rest of the teams can play such that their respective scores do not exceed s_j (which would make team k win). If the maximal flow equals $\sum_{i,k,i\neq k,j\neq k} games(i, j)$, then its possible to to distribute scores such that no team's score exceeds s_i.

For the chess tournament, we have to make a small modification of the graph above. Namely, now each game gives 2 points to the playing teams. Hence just multiply all the capacities by 2, except for the capacities connecting teams with t. Now $s_k = 2\sum_{j\neq k} games(i, j)$ and s_j are defined as previously.

The running time of the algorithm is as follows: there are $2 + n - 1 + n(n-1)/2$ vertices (for s, t, every team and every game) and $n(n-1)/2 + 2n(n-1)/2 + n - 1$ edges, as every game connects exactly 2 teams and there are $(n-1)n/2$ possible games. Hence the running time is $O(n^6)$.

The main difficulty with the football tournament is that we cannot express games as a network flow in the same way, as we don't have flow preservation anymore. In the paper "Football Elimination is Hard to Decide Under the 3-Point Rule" you can find a proof that the problem is actually NP-complete.

Correction: there are $2 + (n-1) + \frac{(n-1)(n-2)}{2}$ vertices

and $\frac{(n-1)(n-2)}{2} + 2\frac{(n-1)(n-2)}{2} + (n-1)$ edges

there are $\frac{(n-1)(n-2)}{2}$ possible games.

Exercises for Unit 27

2) We know that in the network G max-flow = min-cut = k and all edges of some min-cut were destroyed.

ping(v) = true if vertex v is still reachable from s and false otherwise. If we can determine which edges where destroyed, we can report the vertices which are not reachable from s using DFS, without any ping commands.

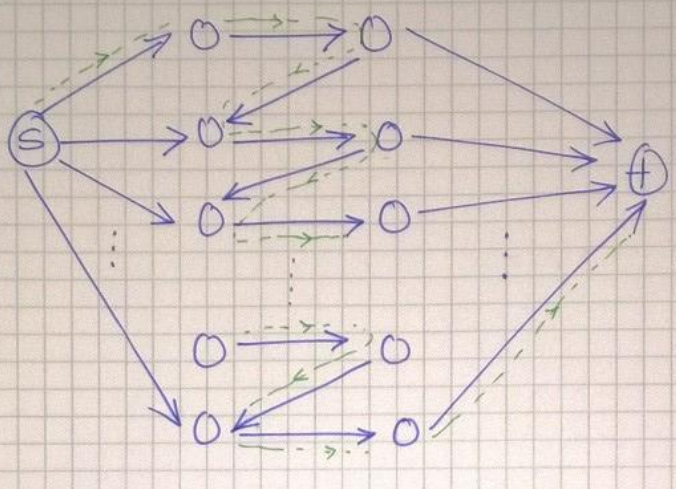
We know that max-flow = k ⇒ ∃ k edge-disjoint s-t paths (we can find them using max-flow algorithm). Each path contains exactly one destroyed edge. Let's consider a path $(s = v_0, v_1, \dots v_n = t)$. If edge (v_i, v_{i+1}) was destroyed ping(v_j) = true if $j \le i$ and false if $j > i$. We use Binary search to determine which edge was destroyed:

```
l=0; r=n';
while (r-l > 1)
{
    m' = ⌊(l+r)/2⌋;
    if (ping(v_m') = true) l=m'; else r=m';
}
return edge (v_l, v_{l+1}) was destroyed
```

We have k paths, each path has a length at most n ⇒ we use at most O(k log n) ping operations.

Exercises for Unit 27

3) One can come up with a counter-example as follows:



→ capacity of all edges is 1

Without reverse edges, one can only send a flow of 1 unit (if the augmenting path is as indicated) whereas the maximum flow can be arbitrarily large.