Algorithms and Data Structures
Winter Term 2015/16
Exercises for Units 29 & 30

1. Let $G = (V, E)$ be an undirected graph with vertices $V = \{v_1, \ldots, v_n\}$. Give an efficient algorithm that, given a sequence of non-negative integers $d_1, \ldots, d_n$, decides in polynomial time if G admits an orientation of the edges such that the outdegree of vertex v_i becomes d_i, for all $i = 1, \ldots, n$, and computes the orientation if it exists.

2. A cycle cover of a directed graph $G = (V, E)$ is a set of vertex-disjoint, simple, directed cycles that cover all the vertices (every vertex included in at least one cycle). Describe and analyze an algorithm to find a cycle cover for a given graph, or correctly report that no cycle cover exists.

3. Suppose you are given an algorithm to compute a minimum-cost perfect matching in any bipartite graph $G = (A \cup B, E)$ with non-negative edge costs $c_e \geq 0$. Show how you can use the algorithm to design algorithms that compute a
   - minimum-cost *maximum* matching,
   - *maximum*-cost perfect matching,
   - minimum-cost perfect matching for *general (possibly negative) edge costs.*

4. In a k-regular undirected graph G every vertex has degree k. Show that every k-regular bipartite graph has a set of k edge-disjoint perfect matchings.

5. Show that an undirected connected tree T has at most one perfect matching. Show that if a tree T has a perfect matching, then for every $v \in V$, the forest $T - \{v\}$ has exactly one component with an odd number of vertices.

6. In a server farm you have m machines, and there are n jobs that should be processed. Job i has processing time $p_{ij} > 0$ if it is processed on machine j. The goal is to assign each job to exactly one machine and for each machine determine an ordering of jobs assigned to it. The machine then processes its assigned jobs one-by-one in the specified order. If job i is put in position k on machine j, the completion time of i is the sum of processing times of jobs at positions $1, \ldots, k$ on machine j. Design and analyze an efficient algorithm to compute an assignment of jobs to machines and orderings such that the sum of all completion times is minimized.