Algorithms and Data Structures
Winter Term 2015/16
Exercises for Unit 5

1. Use the Akkra-Bazi Theorem to derive closed form expressions for the following recursively defined functions:

   a) $T(n) = 2 \cdot T(n/4) + \sqrt{n}$

   b) $T(n) = 3 \cdot T(n/2) + n \log n$

   c) $T(n) = T(n/2) + \alpha$

   d) $T(n) = 4 \cdot T(n/2) + n^2/\log n$

   It is assumed that in all cases we have $T(n) = c \cdot n$ for $n \leq 4$, and α and c are some positive constants.

2. Use the Akkra-Bazi Theorem to derive the "Master Theorem" for divide-and-conquer recurrence relations, which we stated in class and which you find in most textbooks on analysis of algorithms, stated in some form, e.g. as

   Theorem (Master Theorem)

   Let $T(n)$ be a monotonically increasing function that satisfies

   $$\begin{aligned} T(n) &= aT(\tfrac{n}{b}) + f(n) \\ T(1) &= c \end{aligned}$$

   where $a \geq 1, b \geq 2, c > 0$. If $f(n) \in \Theta(n^d)$ where $d \geq 0$, then

   $$T(n) = \begin{cases} \Theta(n^d) & \text{if } a < b^d \\ \Theta(n^d \log n) & \text{if } a = b^d \\ \Theta(n^{\log_b a}) & \text{if } a > b^d \end{cases}$$

3. Prove the other inequality of the AB-Theorem stated in class.

   Let

   $$f(x) = \begin{cases} h(x) & \text{for } 1 \leq x \leq x_0 \\ af(x/b) + g(x) & \text{for } x > x_0. \end{cases}$$

   where

   a) $a > 0$, $b > 1$, and $x_0 \geq b$ are constants,

   b) $x \geq 1$ is a real number,

   c) $d_1 \leq h(x) \leq d_2$ for some positive constants d_1, d_2 and all x with $1 \leq x \leq x_0$, and

   d) g is a nonnegative function satisfying the polynomial growth condition, i.e. there are positive constants c_1 and c_2 such that

   $$c_1 g(x) \leq g(u) \leq c_2 g(x) \quad \text{for all } x > x_0 \text{ and } u \in [x/b, x],$$

   e) and the following technical condition holds:

   $$I := \int_1^{x_0} \frac{g(u)}{u^{p+1}} du < \infty \,.$$

   Let p be the unique real number for which $a/b^p = 1$. Then

   $$f(x) = \Omega\Big(x^p \Big(1 + \int_1^x \frac{g(u)}{u^{p+1}} du\Big)\Big) \,.$$